From Newton via Hamilton to Kepler

Another version of “One Newton yields three Kepler”, based on a paper of Erich Ch. Wittman and the earlier papers of Kepler_0x.pdf.

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\[
\frac{d\varphi}{dt} = \frac{w}{r^2}
\]
1 Orbits in a Central Force Field

If the acting force is everywhere directed to or away from a central point the force is called a central force. The central point is chosen as the zero point of the coordinate frame, and we have $\vec{r} \parallel \vec{a}$ or

$$\vec{r} \times \vec{a} = \vec{0}$$  \hspace{1cm} \text{(1)}

Orbits in the field of a central force lie in a plane. The force acts always in the plane given by the origin and the initial vectors $\vec{r}(t_0)$ and $\vec{v}(t_0)$.

For any movement in the field of a central force we have

$$\vec{r} \times \vec{v} = \vec{w} = \text{konstant}$$  \hspace{1cm} \text{(2)}

Using (1) it is easy to show that the derivative of $\vec{w}$ with respect to time is zero.

$\vec{w}$ is normal to the orbital plane. $\vec{w}$ and $w = |\vec{w}|$ are important invariants of the orbital motion.

$\vec{r} \times (m \cdot \vec{v}) = m \cdot (\vec{r} \times \vec{v}) = m \cdot \vec{w}$ is the angular momentum of the orbiting mass. In a central force field angular momentum is conserved.
The Area swept out by the Position Vector

\[ \Delta A = \frac{1}{2} \cdot |\vec{r} \times \vec{v} \cdot \Delta t| = \frac{1}{2} \cdot \Delta t \cdot |\vec{r} \times \vec{v}| = \frac{1}{2} \cdot \Delta t \cdot w \]

According to (2) we have in any central force field

\[ \frac{dA}{dt} = \frac{1}{2} \cdot w = c = \text{constant} \] (3)

Constant \( c \) is only introduced for better comparison with other scripts. (3) is the essence of Kepler’s Second Law.

Trajectories in a plane can be expressed in polar coordinates:

\[ \Delta A = \frac{1}{2} \cdot r \cdot \Delta b = \frac{1}{2} \cdot r \cdot r \cdot \Delta \varphi \]

For all planar trajectories we have without any further premise

\[ \frac{dA}{d\varphi} = \frac{1}{2} \cdot r^2 \] (4)

From (3) and (4) we get using the chain rule

\[ \frac{d\varphi}{dt} = \frac{w}{r^2} \quad \text{and} \quad \frac{dt}{d\varphi} = \frac{r^2}{w} \] (5)

Proof: The chain rule states

\[ \frac{dA}{dt} = \frac{dA}{d\varphi} \cdot \frac{d\varphi}{dt} \]

and hence

\[ \frac{1}{2} \cdot w = \frac{1}{2} \cdot r^2 \cdot \frac{d\varphi}{dt} \quad \text{and} \quad \frac{w}{r^2} = \frac{d\varphi}{dt} \]
3 Newton’s Law of Gravitation

Following Newton we assume the central force to be spherically symmetric, and the absolute value of the force should be proportional to $\frac{1}{r^2}$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{k}{r^2} \cdot \frac{-\vec{r}}{r}$$

(6)

If a small mass $m$ moves in the gravitational field of a huge mass $M$ we have

$$k = G \cdot M$$

(7)

$G$ denoting Newton’s gravitational constant.

In a spherically symmetric central force field not only angular momentum but also energy is conserved. We will need this fact in section 8.
4 The Hodograph lies on a Circle

According to (6) we have

\[
\frac{d\vec{v}}{dt} = \frac{k}{r^2} \cdot \left( -\frac{\vec{r}}{r} \right) = \frac{k}{r^2} \cdot \left( -\sin \varphi \right)
\]

Using the chain rule and (5) we get

\[
\frac{d\vec{v}}{d\varphi} = \frac{d\vec{v}}{dt} \cdot \frac{dt}{d\varphi} = \frac{k}{r^2} \cdot \left( -\sin \varphi \right) \cdot \frac{r^2}{w} = \frac{k}{w} \cdot \left( -\sin \varphi \right)
\]

Integration with respect to \( \varphi \) gives us the hodograph of the movement:

\[
\vec{v}(\varphi) = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \frac{k}{w} \cdot \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix}
\]

Constant \( h_1 \) is set to zero if we choose the coordinate frame so that \( \vec{r} \) points to the perihelion of the trajectory for \( \varphi = 0 \). \( \vec{r} \) reaches its minimal value only if \( \vec{v} \) is orthogonal to \( \vec{r} \). Then we can write

\[
\vec{v}(\varphi) = \begin{pmatrix} 0 \\ h \end{pmatrix} + \frac{k}{w} \cdot \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix}
\]

(9) is the equation of a circle with center point \( H = (0/h) \) and radius \( \rho \) where

\[
\rho = \frac{k}{w}
\]

The constant of integration \( h \) will be calculated in section 7 and 8.

The ideas of the sections 4 and 5 originate from the following beautiful publication of Erich Ch. Wittmann:

"Von den Hüllkurvenkonstruktionen der Kegelschnitte zu den Planetenbahnen"
5 Orbits in the Field of a Central Mass are Conic Sections

We get another equation for \( \vec{v} \) calculating the derivative of

\[
\vec{r} = r \cdot \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}
\]

with respect to time using (5) and the chain rule:

\[
\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{d\varphi} \cdot \frac{d\varphi}{dt} = \frac{w}{r^2} \cdot \frac{d}{d\varphi} \left( r \cdot \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} \right) = \frac{w}{r^2} \cdot \left( \frac{dr}{d\varphi} \cdot \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} + r \cdot \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix} \right)
\]

(11)

(9) and (11) both give a representation of \( \vec{v}(\varphi) \). For the components of \( \vec{v}(\varphi) \) we get

I \[0 + \frac{k}{w} \cdot (-\sin \varphi) = \frac{w}{r^2} \cdot \frac{dr}{d\varphi} \cdot \cos \varphi + \frac{w}{r} \cdot (-\sin \varphi)\]

II \[h + \frac{k}{w} \cdot \cos \varphi = \frac{w}{r^2} \cdot \frac{dr}{d\varphi} \cdot \sin \varphi + \frac{w}{r} \cdot \cos \varphi\]

Multiplying I by \((-\sin \varphi)\) and II by \(\cos \varphi\) and summation of the new equations yields

\[h \cdot \cos \varphi + \frac{k}{w} = \frac{w}{r}\]

and, multiplying by \(r \cdot w/k\)

\[r \cdot \left( \frac{h \cdot w}{k} \cdot \cos \varphi + 1 \right) = \frac{w^2}{k}\]

(12)

Defining

\[p = \frac{w^2}{k}\]

(13)

and

\[\varepsilon = \frac{h \cdot w}{k}\]

(14)

we get from (12) the equation of a conic section in polar coordinates:

\[r = \frac{p}{1 + \varepsilon \cdot \cos \varphi}\]

(15)

The orbits of the planets are ellipses with the sun in one of their focal points.
In order to derive Kepler’s third law we integrate (3) over a complete period $T$:

$$c \cdot T = \pi \cdot a \cdot b$$

Squared

$$c^2 \cdot T^2 = \pi^2 \cdot a^2 \cdot b^2$$

Using $b^2 = a \cdot p$, (13) and $w^2 = 4 \cdot c^2$ we get

$$\frac{a^3}{T^2} = \frac{c^2}{p \cdot \pi^2} = \frac{c^2 \cdot k}{4 \cdot c^2 \cdot \pi^2} = \frac{k}{4 \cdot \pi^2} = \frac{G \cdot M}{4 \cdot \pi^2}$$  \hspace{1cm} (16)

The quotient $a^3/T^2$ takes the same value for all planets. We have got Kepler’s third law.

Now, Newton’s third law states that all forces between bodies are interactions, abbreviated oftly by *actio = reactio*. The sun has to move around the common center of mass of $M$ und $m$, too. This leads to a small correction of (16). The exact version of Kepler’s third law within Newton’s theory of gravitation is given by

$$\frac{a^3}{T^2} = \frac{G \cdot (M + m)}{4 \cdot \pi^2}$$  \hspace{1cm} (17)

(17) is symmetric as $M$ and $m$ are concerned. In order to calculate the masses of the components of a binary star you have to use (17). The formula is derived in Kepler_01.pdf. Within our solar system (17) is a very small refinement of (16).

All the papers Kepler_xy.pdf are offered for download at www.physastromath.ch/material/mathematik/keplernewton/
7 Excentricity, total Energy and the Center of the Hodograph

With (10) and (13) we have for the radius of the hodograph

\[ \rho = \frac{k}{w} = \frac{G \cdot M}{w} = \frac{w}{\rho} \]  \hspace{1cm} (18)

From (14) we get for the constant of integration \( h \)

\[ h = \epsilon \cdot \frac{k}{w} = \epsilon \cdot \rho \] \hspace{1cm} (19)

So we have

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Ellipse</th>
<th>Parabola</th>
<th>Hyperbola</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excentricity</td>
<td>( \epsilon &lt; 1 )</td>
<td>( \epsilon = 1 )</td>
<td>( \epsilon &gt; 1 )</td>
</tr>
<tr>
<td>total Energy</td>
<td>( E_{\text{tot}} &lt; 0 )</td>
<td>( E_{\text{tot}} = 0 )</td>
<td>( E_{\text{tot}} &gt; 0 )</td>
</tr>
<tr>
<td>( \rho ) and ( h )</td>
<td>( h &lt; \rho )</td>
<td>( h = \rho )</td>
<td>( h &gt; \rho )</td>
</tr>
<tr>
<td>( \rho ) and ( v_p )</td>
<td>( v_p &lt; 2 \cdot \rho )</td>
<td>( v_p = 2 \cdot \rho )</td>
<td>( v_p &gt; 2 \cdot \rho )</td>
</tr>
<tr>
<td>Position of ( O )</td>
<td>in the hodograph</td>
<td>on the hodograph</td>
<td>outside of the hodograph</td>
</tr>
</tbody>
</table>

Let’s draw the hodographs in all three cases. \( \vec{v}_p = \overrightarrow{OP} \) is the velocity of \( m \) in the perihelion, that is the maximum velocity of \( m \).

For \( \epsilon < 1 \) the center \( O \) is within the hodograph:
For $\varepsilon > 1$ the center $O$ is exterior to the circle of the hodograph:

![Diagram of velocity vectors and hodograph]

The tips of the velocity vectors all lie on the arc $APB$. $A$ and $B$ are reached at distance $r = \infty$ from the sun only.

Let’s prove the following small proposition: $A$ and $B$ are points of the Thales circle with diameter $OH$.

**Proof:** $\varphi_{\text{max}}$ is reached at $r = \infty$. Then we have according to (15)

$$1 + \varepsilon \cdot \cos \varphi_{\text{max}} = 0$$

rearranged

$$\frac{1}{\varepsilon} = -\cos \varphi_{\text{max}}$$

or

$$\frac{\rho}{h} = -\cos \varphi_{\text{max}} = \cos(180^\circ - \varphi_{\text{max}})$$

This holds if and only if $A$ and $B$ are points of the Thales circle with diameter $OH$:

$$\cos(180^\circ - \varphi_{\text{max}}) = \cos \alpha = \frac{BT}{OH} = \frac{\rho}{h}$$
For $\varepsilon = 1$ the hodograph looks as follows:

![Diagram of a hodograph](image)

All points of the circle are covered but $O$. Associated to $O$ is the distance $r = \infty$ from the sun. $|\overline{OP}| = 2 \cdot \rho$ is the maximum velocity in the perihelion.
8 Calculating the Orbital Elements from $\mathbf{r}(t_0)$ and $\mathbf{v}(t_0)$

Let $\mathbf{r}$ and $\mathbf{v}$ be known for any specific moment. Then we get from (2) the constants $w$ and $c = \frac{1}{2} \cdot w$.

The central mass $M$ gives us $k = G \cdot M$. With (10) we get the radius $\rho$ of the hodograph, and from (13) we know the semi-latus rectum $p$ of the orbit.

Now we use once more Newton’s law of gravitation. The gravitational field of $M$ is “conservativ”, total energy is conserved:

$$\frac{1}{2} m \cdot v^2 - G \cdot M \cdot m \cdot \frac{1}{r} = E_{\text{tot}} = -G \cdot M \cdot m \cdot \frac{1}{2} \cdot a$$  \hspace{1cm} (20)

or

$$v^2 - \frac{2 \cdot k}{r} = - \frac{k}{a}$$  \hspace{1cm} (21)

$2 \cdot a$ is the radius of the “cercle directeur” (in German: “Leitkreis”) of the conic section (15). In case of an ellipse $a$ is the semi-major axis of the ellipse! You find more on this in Kepler\_09.pdf!

Solving (21) with respect to $a$ we get

$$a = \frac{k \cdot r}{2 \cdot k - v^2 \cdot r}$$  \hspace{1cm} (22)

The values of $\varepsilon$ and $h$ are still missing. (19) tells us that knowing $\varepsilon$ means knowing $h$ too. With $p = a \cdot (1 - \varepsilon^2)$ and (13) we get from (22)

$$\varepsilon^2 = 1 - \frac{p}{a} = 1 - \frac{2 \cdot w^2}{k \cdot r} + \frac{v^2 \cdot w^2}{k^2}$$  \hspace{1cm} (23)

(23) determines $\varepsilon$ and $h = \varepsilon \cdot \rho$.

(20) to (23) are equally valid for hyperbolas, with $a$ having a negative sign. For parabolas $a$ gets the value infinity from (20). However, for parabolas we always have $\varepsilon = 1$!

Using (10) and (13) we can rewrite (23) as follows:

$$\varepsilon^2 = 1 - 2 \cdot \frac{p}{r} + \frac{v^2}{\rho^2}$$  \hspace{1cm} (24)

(24) shows clearly that $\varepsilon$ is a pure number. $p$ and $r$ both are lengths, $v$ and $\rho$ both are velocities.