

Teaching the Kepler Laws for Freshmen

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One of the highlights of classical mechanics is the mathematical derivation of the three experimentally observed Kepler laws of planetary motion from Newton's laws of motion and of gravitation. Newton published his theory of gravitation in 1687 in the *Principia Mathematica* [13]. After two short introductions, one with definitions and the other with axioms (the laws of motion), Newton discussed the Kepler laws in the first three sections of Book 1 (in just 40 pages, without ever mentioning the name of Kepler!).

Kepler's second law (motion is planar and equal areas are swept out in equal times) is an easy consequence of the conservation of angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and holds in greater generality for any central force field. All this is explained well by Newton in Propositions 1 and 2.

On the other hand, Kepler's first law (planetary orbits are ellipses with the center of the force field at a focus) is specific for the attractive $1/r^2$ force field. Using Euclidean geometry, Newton derives in Proposition 11 that the Kepler laws can hold only for an attractive $1/r^2$ force field. The converse statement, that an attractive $1/r^2$ force field leads to elliptical orbits, Newton concludes in Corollary 1 of Proposition 13. Tacitly he assumes for this argument that the equation of motion $\mathbf{F} = m\mathbf{a}$ has a unique solution for given initial position and initial velocity. Theorems about existence and uniqueness of solutions of such a differential equation were formulated and mathematically proven only in the 19th century. However, there can be little doubt that Newton did grasp these properties of his equation $\mathbf{F} = m\mathbf{a}$ [1].

Later, in 1710, Jakob Hermann and Johan Bernoulli gave a direct proof of Kepler's first law, which is still the standard proof for modern textbooks on classical mechanics [14]. One writes the position vector \mathbf{r} in the plane of motion in polar coordinates r and θ . The trick is to transform the equation of motion $m\mathbf{a} = -k\mathbf{r}/r^3$ with variable the time t into a second-order differential equation of the scalar function $u = 1/r$ with variable the angle θ . This differential

equation can be solved exactly, and yields the equation of an ellipse in polar coordinates [4].

Another popular proof goes by writing down the so-called Runge-Lenz vector

$$\mathbf{K} = \mathbf{p} \times \mathbf{L} - km\mathbf{r}/r$$

with $\mathbf{p} = m\mathbf{v}$ the momentum and $\mathbf{F}(\mathbf{r}) = -k\mathbf{r}/r^3$ the force field of the Kepler problem. The Runge-Lenz vector \mathbf{K} turns out to be conserved, i.e., $\dot{\mathbf{K}} = \mathbf{0}$. This result can be derived by a direct computation as we indicate in the next section. An alternative geometric argument is sketched in the following section. Working out the equation $\mathbf{r} \cdot \mathbf{K} = rK \cos \theta$ yields the equation of an ellipse in polar coordinates [4]. The geometric meaning of the Runge-Lenz vector becomes clear *a posteriori*: it is a vector pointing in the direction of the major axis of the ellipse. But at the start of the proof, writing down the Runge-Lenz vector seems an unmotivated trick.

For an historical account of the Runge-Lenz vector, we refer to Goldstein [5, 6]. Goldstein traces the vector back to Laplace's *Traité de Mécanique Céleste* from 1798. Actually, the Runge-Lenz vector already appeared in a paper by Lagrange from 1781 [9], which as far as we know was the vector's first use. Lagrange writes the vector down after algebraic manipulations and without any geometric motivation.

It is more than clear by now that the name Runge-Lenz vector is inappropriate, but with its widespread use in modern literature it seems too late to change that.

The purpose of this note is to present in the first section a proof of the Kepler laws for which *a priori* the reasoning is well motivated in both physical and geometric terms. Then, in the following section, we review the hodographic proof as given by Feynman in his "Lost Lecture" [7], and finally we discuss Newton's proof from the *Principia* [13]. All three proofs are based on Euclidean geometry, although we do use the language of vector calculus in order to make the text more readable for people of the 21st century. We feel that our proof is really the simplest of the three, and at the same time it gives more refined information (namely

the length of the major axis $2a = -k/H$ of the ellipse \mathcal{E}). In fact we think that our proof in the next section can compete both in transparency and in level of computation with the standard proof of Jakob Hermann and Johann Bernoulli, making it an appropriate alternative to present in a freshman course on classical mechanics.

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A Euclidean Proof of Kepler's First Law

We shall use inner (or scalar, or dot) products $\mathbf{u} \cdot \mathbf{v}$ and outer (or vector, or cross) products $\mathbf{u} \times \mathbf{v}$ of vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^3 , the compatibility conditions

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w},$$

and the Leibniz product rules

$$(\mathbf{u} \cdot \mathbf{v})' = \dot{\mathbf{u}} \cdot \mathbf{v} + \mathbf{u} \cdot \dot{\mathbf{v}}$$

$$(\mathbf{u} \times \mathbf{v})' = \dot{\mathbf{u}} \times \mathbf{v} + \mathbf{u} \times \dot{\mathbf{v}}$$

without further explanation.

For a central force field $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{r}/r$ the angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved by Newton's law of motion $\mathbf{F} = \dot{\mathbf{p}}$, thereby leading to Kepler's second law. For a spherically symmetric central force field $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{r}/r$, the energy

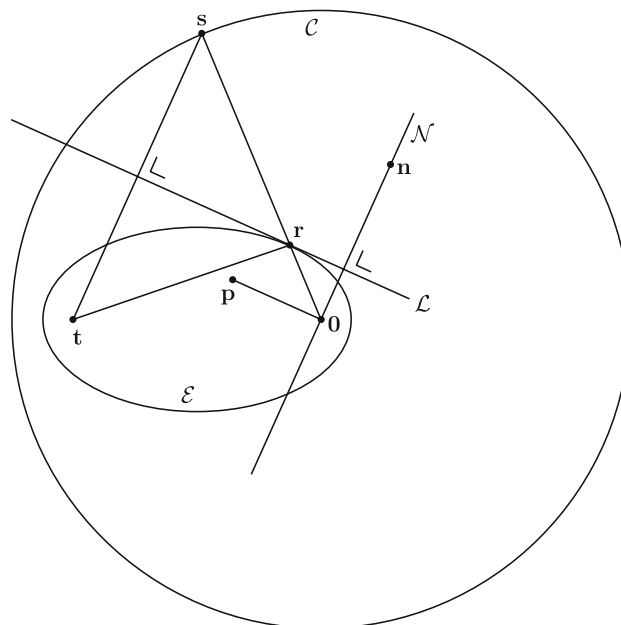
$$H = p^2/2m + V(r), \quad V(r) = - \int f(r)dr$$

is conserved as well. These are the general initial remarks.

From now on, consider the Kepler problem $f(r) = -k/r^2$ and $V(r) = -k/r$, with $k > 0$ a coupling constant. If m is replaced by the reduced mass $\mu = mM/(m + M)$ then the coupling constant becomes $k = GmM$, with m and M the masses of the two bodies and G the universal gravitational constant. Using conservation of energy, we show that the

motion for fixed energy $H < 0$ is bounded inside a sphere with center $\mathbf{0}$ and radius $-k/H$. Indeed, $V(r) \leq H$, and so $k/r \geq -H$ or equivalently $r \leq -k/H$.

Consider the following picture of the plane perpendicular to \mathbf{L} .



The circle \mathcal{C} with center $\mathbf{0}$ and radius $-k/H$ is the boundary of a disc where motion with energy $H < 0$ takes place. Points that fall from the circle \mathcal{C} have the same energy as the original moving point, and for this reason \mathcal{C} is called the fall circle. Let $\mathbf{s} = -k\mathbf{r}/rH$ be the projection of \mathbf{r} from the center $\mathbf{0}$ onto the fall circle \mathcal{C} . The line \mathcal{L} through \mathbf{r} with direction vector \mathbf{p} is the tangent line of the orbit \mathcal{E} at position \mathbf{r} with velocity \mathbf{v} . Let \mathbf{t} be the orthogonal reflection of the point \mathbf{s} in the line \mathcal{L} . As time varies, the position vector \mathbf{r} moves along the orbit \mathcal{E} , and likewise \mathbf{s} moves along the fall circle \mathcal{C} . It is good to investigate how the point \mathbf{t} moves.



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THEOREM. The point \mathbf{t} equals \mathbf{K}/mH and therefore is conserved.

PROOF. The line \mathcal{N} spanned by $\mathbf{n} = \mathbf{p} \times \mathbf{L}$ is perpendicular to \mathcal{L} . The point \mathbf{t} is obtained from \mathbf{s} by subtracting twice the orthogonal projection of $\mathbf{s} - \mathbf{r}$ on the line \mathcal{N} , and therefore

$$\mathbf{t} = \mathbf{s} - 2((\mathbf{s} - \mathbf{r}) \cdot \mathbf{n})\mathbf{n}/n^2.$$

Now

$$\mathbf{s} = -k\mathbf{r}/rH$$

$$(\mathbf{s} - \mathbf{r}) \cdot \mathbf{n} = -(H + k/r)\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L})/H = -(H + k/r)L^2/H$$

$$n^2 = p^2L^2 = 2m(H + k/r)L^2,$$

and therefore

$$\mathbf{t} = -k\mathbf{r}/rH + \mathbf{n}/mH = \mathbf{K}/mH,$$

where $\mathbf{K} = \mathbf{p} \times \mathbf{L} - km\mathbf{r}/r$ is the Runge-Lenz vector. The final step $\dot{\mathbf{K}} = \mathbf{0}$ is derived by a straightforward computation, using the compatibility relations and the Leibniz product rules for inner and outer products of vectors in \mathbb{R}^3 .

COROLLARY. The orbit \mathcal{E} is an ellipse with foci $\mathbf{0}$ and \mathbf{t} , and major axis equal to $2a = -k/H$.

PROOF. Indeed we have

$$|\mathbf{t} - \mathbf{r}| + |\mathbf{r} - \mathbf{0}| = |\mathbf{s} - \mathbf{r}| + |\mathbf{r} - \mathbf{0}| = |\mathbf{s} - \mathbf{0}| = -k/H.$$

Hence \mathcal{E} is an ellipse with foci $\mathbf{0}$ and \mathbf{t} , and major axis $2a = -k/H$.

The above proof has two advantages over the earlier mentioned proofs of Kepler's first law. The conserved vector $\mathbf{t} = \mathbf{K}/mH$ is *a priori* well motivated in geometric terms. Moreover we use the gardener's definition of an ellipse. The gardener's definition, so called because gardeners sometimes use this construction for making an oval flowerbed, is well known to (Dutch) freshmen. In contrast, the equation of an ellipse in polar coordinates is unknown to most freshmen, and so additional explanation would be needed for that. Yet another advantage of our proof is that the solution of the equation of motion is achieved by just finding enough constants of motion (of geometric origin), whose integration is performed trivially by the fundamental theorem of calculus. The proofs by Feynman and Newton in the next sections on the contrary rely at a crucial point on the existence and uniqueness theorem for differential equations.

We proceed to derive Kepler's third law along standard lines [4]. The ellipse \mathcal{E} has numerical parameters (the major axis equals $2a$, the minor axis $2b$ and $a^2 = b^2 + c^2$) $a, b, c > 0$ given by $2a = -k/H$, $4c^2 = K^2/m^2H^2 = (2mHL^2 + m^2k^2)/m^2H^2$. The area of the region bounded by \mathcal{E} equals

$$\pi ab = LT/2m,$$

with T the period of the orbit. Indeed, $L/2m$ is the area of the sector swept out by the position vector \mathbf{r} per unit time. A straightforward calculation yields

$$T^2/a^3 = 4\pi^2m/k.$$

The mass m we have used so far is actually equal to the reduced mass $\mu = mM/(m + M)$, with m the mass of the planet and M the mass of the sun, and this almost equals m if $m \ll M$. The coupling constant k is, according to Newton, equal to GmM with G the universal gravitational constant. We therefore see that Kepler's (harmonic) third law, stating that T^2/a^3 is the same for all planets, holds only approximately for $m \ll M$.

It might be a stimulating question for the students to adapt the arguments of this section to the case of fixed energy $H > 0$. Under this assumption, the motion becomes unbounded and traverses one branch of a hyperbola.

Feynman's Proof of Kepler's First Law

In this section we discuss a different geometric proof of Kepler's first law based on the hodograph \mathcal{H} . By definition \mathcal{H} is the curve traced out by the velocity vector \mathbf{v} in the Kepler problem. This proof goes back to Möbius in 1843 and Hamilton in 1845 [3] and has been forgotten and rediscovered several times, by Maxwell in 1877 [2] and by Feynman in 1964 in his "Lost Lecture" [7], among others.

Let us assume (as in the picture in the previous section) that $i\mathbf{v}\mathbf{n}/n = \mathbf{v}$, with i the counterclockwise rotation around $\mathbf{0}$ over $\pi/2$. So the orbit \mathcal{E} is assumed to be traversed counterclockwise around the origin $\mathbf{0}$.

THEOREM. The hodograph \mathcal{H} is a circle with center $\mathbf{c} = i\mathbf{K}/mL$ and radius k/L .

PROOF. We shall indicate two proofs of this theorem. The first proof is analytic in nature, and uses conservation of the Runge-Lenz vector \mathbf{K} by rewriting

$$\mathbf{K} = \mathbf{p} \times \mathbf{L} - km\mathbf{r}/r = mv\mathbf{L}\mathbf{n}/n - km\mathbf{r}/r$$

as

$$v\mathbf{n}/n = \mathbf{K}/mL + k\mathbf{r}/rL,$$

or equivalently

$$\mathbf{v} = i\mathbf{K}/mL + ik\mathbf{r}/rL.$$

Hence the theorem follows from $\dot{\mathbf{K}} = \mathbf{0}$.

There is a different geometric proof of the theorem, discussed by Feynman, which, instead of using the conservation of the Runge-Lenz vector \mathbf{K} , yields it as a corollary. The key point is to reparametrize the velocity vector \mathbf{v} from time t to angle θ of the position vector \mathbf{r} . It turns out that the vector $\mathbf{v}(\theta)$ is traversing the hodograph \mathcal{H} with constant speed k/L . Indeed we have from Newton's laws

$$m \frac{d\mathbf{v}}{dt} = m\mathbf{a} \frac{dt}{d\theta} = -\frac{k\mathbf{r}}{r^3} \frac{dt}{d\theta},$$

and Kepler's second law yields

$$r^2 d\theta/2 = Ldt/2m.$$

Combining these identities yields

$$\frac{d\mathbf{v}}{d\theta} = -k\mathbf{r}/rL,$$

so indeed $\mathbf{v}(\theta)$ travels along \mathcal{H} with constant speed k/L . Since $\mathbf{r} = re^{i\theta}$, a direct integration yields

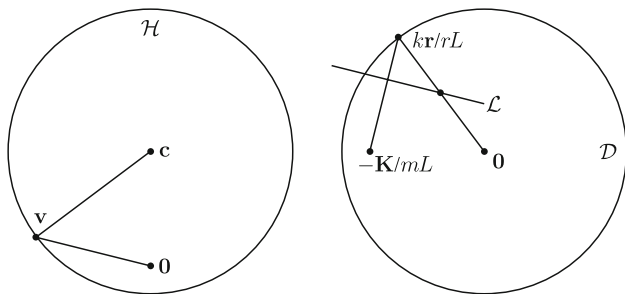
$$\mathbf{v}(\theta) = \mathbf{c} + ikr/rL, \quad \dot{\mathbf{c}} = \mathbf{0},$$

and the hodograph becomes a circle with center \mathbf{c} and radius k/L . Comparison with the last formula in the first proof gives

$$\mathbf{c} = i\mathbf{K}/mL,$$

and $\dot{\mathbf{K}} = \mathbf{0}$ comes out as a corollary.

All in all, the circular nature of the hodograph \mathcal{H} is more or less equivalent to the conservation of the Runge-Lenz vector \mathbf{K} .



Now turn the hodograph \mathcal{H} clockwise around $\mathbf{0}$ by $\pi/2$ and translate by $i\mathbf{c} = -\mathbf{K}/mL$. This gives a circle \mathcal{D} with center $\mathbf{0}$ and radius k/L . Since

$$kr/rL + \mathbf{K}/mL = \mathbf{v}n/n = -i\mathbf{v},$$

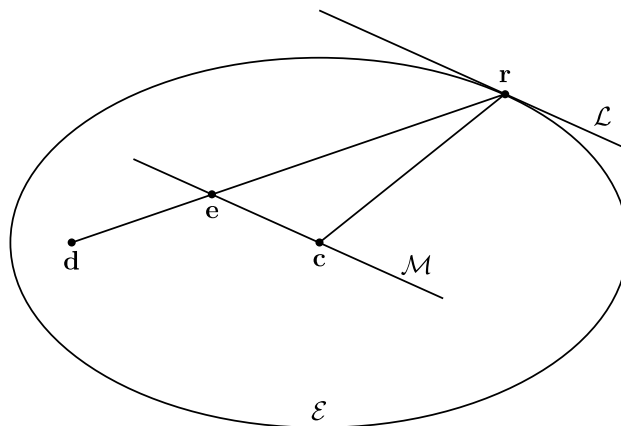
the orbit \mathcal{E} intersects the line through $\mathbf{0}$ and kr/rL in a point with tangent line \mathcal{L} perpendicular to the line through kr/rL and $-\mathbf{K}/mL$. For example, the ellipse \mathcal{F} with foci $\mathbf{0}$ and $-\mathbf{K}/mL$ and major axis equal to k/L has this property, but any scalar multiple $\lambda\mathcal{F}$ with $\lambda > 0$ has the property as well. Because curves with the above property are uniquely characterized after an initial point on the curve is chosen, we conclude that $\mathcal{E} = \lambda\mathcal{F}$ for some $\lambda > 0$. This proves Kepler's first law. A comparison with the picture in the previous section shows that $\mathcal{E} = \lambda\mathcal{F}$ with $\lambda = -L/H$. Indeed, \mathcal{E} has foci $\mathbf{0}$ and $-\lambda\mathbf{K}/mL = \mathbf{K}/mH = \mathbf{t}$, and its major axis is equal to $\lambda k/L = -k/H = 2a$.

It is not clear to us whether Feynman was aware that he was relying on the existence and uniqueness theorem for differential equations. On page 164 of [7] the authors quote Feynman: "Therefore, the solution to the problem is an ellipse - or the other way around, really, is what I proved: that the ellipse is a possible solution to the problem. And it is this solution. So the orbits are ellipses."

Apparently Feynman had trouble following Newton's proof of Kepler's first law. On page 111 of [7] the authors write, "In Feynman's lecture, this is the point at which he finds himself unable to follow Newton's line of argument any further, and so sets out to invent one of his own".

Newton's Proof of Kepler's First Law

In this section we discuss a modern version of the original proof by Newton of Kepler's first law as given in [13]. The proof starts with a nice general result.



THEOREM. Let \mathcal{E} be a smooth closed curve bounding a convex region containing two points \mathbf{c} and \mathbf{d} . Let $\mathbf{r}(t)$ traverse the curve \mathcal{E} counterclockwise in time t , such that the areal speed with respect to the point \mathbf{c} is constant. Likewise let $\mathbf{r}(s)$ traverse the curve \mathcal{E} counterclockwise in time s , such that the areal speed with respect to the point \mathbf{d} is equal to the same constant.

Let \mathcal{L} be the tangent line to \mathcal{E} at the point \mathbf{r} , and let \mathbf{e} be the intersection point of the line \mathcal{M} , which is parallel to \mathcal{L} through the point \mathbf{c} , and the line through the points \mathbf{r} and \mathbf{d} . Then the ratio of the two accelerations is given by

$$\left| \frac{d^2\mathbf{r}}{ds^2} \right| : \left| \frac{d^2\mathbf{r}}{dt^2} \right| = |\mathbf{r} - \mathbf{e}|^3 : (|\mathbf{r} - \mathbf{c}| \cdot |\mathbf{r} - \mathbf{d}|^2).$$

PROOF. Using the chain rule, we get

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{ds}$$

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{d^2\mathbf{r}}{dt^2} \cdot \left(\frac{dt}{ds} \right)^2 + \frac{d\mathbf{r}}{dt} \cdot \frac{d^2t}{ds^2}.$$

Because $d^2\mathbf{r}/dt^2$ is proportional to $\mathbf{c} - \mathbf{r}$ and likewise $d^2\mathbf{r}/ds^2$ is proportional to $\mathbf{d} - \mathbf{r}$, we see that

$$\begin{aligned} \left| \frac{d^2\mathbf{r}}{ds^2} \right| : \left| \frac{d^2\mathbf{r}}{dt^2} \right| &= \left(\frac{dt}{ds} \right)^2 \cdot \left| \frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{r}}{dt} \cdot \frac{d^2t}{ds^2} \right| / \left(\frac{dt}{ds} \right)^2 : \left| \frac{d^2\mathbf{r}}{dt^2} \right| \\ &= \left(\frac{dt}{ds} \right)^2 \cdot |\mathbf{r} - \mathbf{e}| : |\mathbf{r} - \mathbf{c}|. \end{aligned}$$

Since the curve \mathcal{E} is traversed with equal areal speed relative to the two points \mathbf{c} and \mathbf{d} , we get

$$\left| \frac{d\mathbf{r}}{dt} \right| \cdot |\mathbf{r} - \mathbf{e}| = \left| \frac{d\mathbf{r}}{ds} \right| \cdot |\mathbf{r} - \mathbf{d}|$$

and therefore also

$$\frac{dt}{ds} = |\mathbf{r} - \mathbf{e}| : |\mathbf{r} - \mathbf{d}|.$$

In turn this implies that

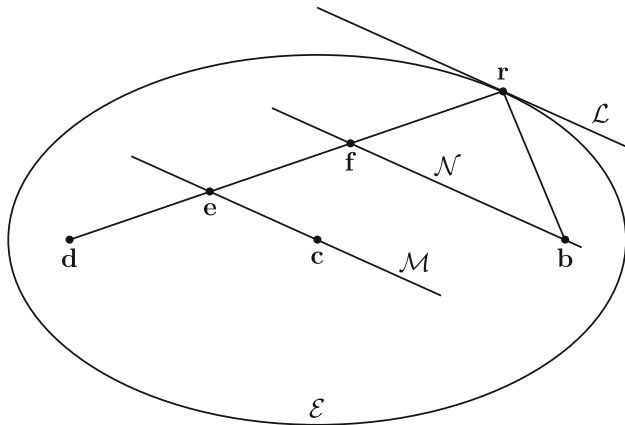
$$\begin{aligned} \left| \frac{d^2\mathbf{r}}{ds^2} \right| : \left| \frac{d^2\mathbf{r}}{dt^2} \right| &= \left(\frac{dt}{ds} \right)^2 \cdot |\mathbf{r} - \mathbf{e}| : |\mathbf{r} - \mathbf{c}| \\ &= |\mathbf{r} - \mathbf{e}|^3 : (|\mathbf{r} - \mathbf{c}| \cdot |\mathbf{r} - \mathbf{d}|^2), \end{aligned}$$

which proves the theorem.

We shall apply this theorem in case \mathcal{E} is an ellipse with center \mathbf{c} and focus \mathbf{d} . Assume that $\mathbf{r}(t)$ traverses the ellipse \mathcal{E} in harmonic motion, say

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{c} - \mathbf{r},$$

so the period for time t is assumed to be 2π .



Let \mathbf{b} be the other focus of \mathcal{E} , and let \mathbf{f} be the intersection point of the line \mathcal{N} , passing through \mathbf{b} and parallel to \mathcal{L} , with the line through the points \mathbf{d} and \mathbf{r} . Then we find

$$|\mathbf{d} - \mathbf{e}| = |\mathbf{e} - \mathbf{f}|, |\mathbf{f} - \mathbf{r}| = |\mathbf{b} - \mathbf{r}|,$$

which in turn implies that $|\mathbf{e} - \mathbf{r}|$ is equal to the half major axis a of the ellipse \mathcal{E} . We conclude from the formula in the previous theorem that the motion in time s along an ellipse with constant areal speed with respect to a focus is only possible in an attractive inverse-square force field. The converse statement, that an inverse-square force field (for negative energy H) indeed yields ellipses as orbits, follows from existence and uniqueness theorems for solutions of Newton's equation $\mathbf{F} = m\mathbf{a}$ and the previously mentioned reasoning. This is Newton's line of argument for proving Kepler's first law.

Conclusion

There exist other proofs of Kepler's law of ellipses from a higher viewpoint. One such proof by Arnold uses complex analysis, and is somewhat reminiscent of Newton's previously described proof by comparing harmonic motion with motion under an $1/r^2$ force field [1]. Apparently Kasner had discovered the same method already back in 1909 [12]. Another proof, by Moser, is also elegant, and uses the language of symplectic geometry and canonical transformations [8, 10, 11]. However our goal here has been to present a proof that is as basic as possible, and at the same time is well motivated in terms of Euclidean geometry.

It is difficult to exaggerate the importance of the role of the *Principia Mathematica* in the history of science.

The year 1687 marks the birth of both modern mathematical analysis and modern theoretical physics. As such, the derivation of the Kepler laws from Newton's law of motion and law of universal gravitation is a rewarding subject to teach to freshmen students. In fact, this was the motivation for our work: we plan to teach this material to high-school students in their final year. Of course, the high-school students first need to become acquainted with the basics of vector geometry and vector calculus. But after this familiarity is achieved, nothing else hinders the understanding of our proof of Kepler's law of ellipses.

For freshmen physics or mathematics students in the university, who are already familiar with vector calculus, our proof given here is fairly short and geometrically well motivated. In our opinion, of all proofs, this proof qualifies best to be discussed in an introductory course.

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