'Half of the Speed' in Relativity

Martin Gubler, Gymnasium Frauenfeld, Frauenfeld, Switzerland

0. Starting point

In a elementary course on STR you have proven the theorem of the addition of velocities parallel to the relative motion of two systems S and S'

$$u = v \oplus u' = \frac{v + u'}{1 + \frac{v}{c} \cdot \frac{u'}{c}} \qquad (1)$$

where v is the velocity of S' relative to S, u' is the speed of some object X measured in S' and u is the speed of this same object X measured in S.

1. 'half of the speed'

I usually leave it as an exercise to the students to find the 'half speed' w with $w \oplus w = v$. To do so they have to solve a quadratic equation (or they let the CAS on their pocket calculator do the job ...).
The solution is

$$w = \frac{v}{1 + \sqrt{1 - v^2 / c^2}} \qquad (2)$$

As the velocitiy v decreases the root term approaches 1 and we have the 'classic' solution $w = v / 2$. Strangely this result seems to be widely unknown. The systems S and S' are moving in a completely symmetric way with relative velocities of -w and w if observed from a third system S'' moving with w relative to S.

In the following two pieces I would like to demonstrate the usefulness of this 'half speed' w. In section 2 we can avoid a lot of algebra using (2), and in section 3 we will establish new relations betweeen the 'classic' formula for the kinetic energy and the corresponding formula in STR.

2. Max Born[1] was probably the first to look at a completely inelastic collision of two bodies of the same restmass to derive the speed dependency of mass. Modern presentations of this idea are to be found in Sartori[2] and Adams[3], e.g.

In a system S two bodies move towards each other with momenta $m_w \cdot w$ and $m_w \cdot (-w)$ before their inelastic collision. $m_w = m(w)$ denotes the mass of the body when its speed is w. The total momentum is zero, and so, after the collision, we have a single mass $M = M_0$ at rest.

Now consider the same collision observed from a system S' moving with -w relative to S. In S', body 2 is at rest and body 1 moves with $v = w \oplus w$ before the collision. After the collision, the velocity of M is w. We write down the equations for the conservation of momentum and the conservation of mass as observed in S' :

$$m_v \cdot v = M_w \cdot w \qquad (3)$$

$$m_v + m_0 = M_w \qquad (4)$$

Using equation (2) and (4) to substitute w and M_w on the right side of (3) we immediately have

$$m_v \cdot \sqrt{1-\frac{v^2}{c^2}} = m_0 \qquad (5)$$

The algebraic effort is much greater if you substitute v by $w \oplus w$ on the left side of (3) (cf [1,2,3]).

3. It is far from being obvious that the relativistic expression for the kinetic energy

$$E_{kin} = m_0 \cdot c^2 \cdot \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \right) = (m_v - m_0) \cdot c^2 \qquad (6)$$

approaches

$$E_{kin} = \frac{1}{2} \cdot m_0 \cdot v^2 \qquad (7)$$

if v is much smaller than c . One way to demonstrate this fact is to develop the bracket term of (6) into a power series and then to strip the terms of higher order. But there is no need to use this heavy calculus tool, and we can even win deeper insights by means of our equation (2).

Let us start with the equation that relates the total energy E_{tot}, the rest energy E_0 and the momentum p of a particle:

$$E_{tot}^2 = E_0^2 + (p \cdot c)^2 \qquad (8)$$

This is the Pythagorean theorem for the corresponding Epstein diagram [4,5]. (8) is equivalent to

$$m_v^2 \cdot c^4 = m_0^2 \cdot c^4 + m_v^2 \cdot v^2 \cdot c^2 \qquad (9)$$

Dividing by c^2 and rearranging the terms we get

$$\left(m_v^2 - m_0^2\right) \cdot c^2 = m_v^2 \cdot v^2 \qquad (10)$$

Dividing by $(m_v + m_0)$ the left side yields the STR expression for the kinetic energy:

$$\left(m_v - m_0\right) \cdot c^2 = \frac{m_v^2}{m_v + m_0} \cdot v^2 = \frac{m_v}{1 + m_0 / m_v} \cdot v^2 \qquad (11)$$

Now we use (5) and (2) to develop (11) to

$$E_{kin} = \frac{m_v}{1+\sqrt{1-v^2/c^2}} \cdot v^2 = m_v \cdot v \cdot \frac{v}{1+\sqrt{1-v^2/c^2}} = p \cdot w \qquad (12)$$

The last representation parallels the classical term

$$E_{kin} = \frac{1}{2} \cdot m_0 \cdot v^2 = m_0 \cdot v \cdot \frac{v}{2} = p \cdot w \qquad (13)$$

In both cases we have $E_{kin} = p \cdot w$, where p stands for the momentum and w for the 'half speed'! Further we have the relations

$$p_{STR} = m_v \cdot v = m_0 \cdot v \cdot \frac{1}{\sqrt{1 - v^2 / c^2}} = p_{CLASSIC} \cdot \frac{1}{\sqrt{1 - v^2 / c^2}} \qquad (14)$$

$$w_{STR} = \frac{v}{1 + \sqrt{1 - v^2 / c^2}} = \frac{v}{2} \cdot \frac{2}{1 + \sqrt{1 - v^2 / c^2}} = w_{CLASSIC} \cdot \frac{2}{1 + \sqrt{1 - v^2 / c^2}} \qquad (15)$$

Now it is obvious that the STR expression for the kinetic energy comes close to the classic term if v is much smaller than the speed of light ! And we certainly do not get the STR expression just by replacing m_0 by m_v as sometimes suggested by students. Both factors p and w are changing when STR enters the stage.

References

1. Max Born, *Die Relativitätstheorie Einsteins* (Springer Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo 1920[1] und 1969[5]), section 7

2. Leo Sartori, *Understanding Relativity* (Univ. of California Press, Berkeley, Los Angeles, London, 1996), section 7.3

3. Steve Adams, *Relativity - An introduction to space-time physics* (Taylor & Francis, Boca Raton, London, New York, 1997), section 2.12.2

4. Lewis C. Epstein, *Relativity visualized* (Insight Press, San Francisco, 1983), chapter 7 & 8

5. David Eckstein on "www.relativity.li/Epstein/" section 6.5

Martin Gubler teaches Math and Physics at the Gymnasium Frauenfeld, Switzerland. He graduated from the University of Zurich in 1979 and earned his PhD in 1983 in group theory. He has since been teaching with special interests in astronomy and relativity. He is the head behind the website "www.relativity.li". For US-readers: The last year of a German or Swiss 'Gymnasium' compares to the first year of a US college.

Kantonsschule Frauenfeld, Ringstrasse 10, CH - 8500 Frauenfeld, Switzerland, gub@bluewin.ch

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